A Monoid Structure on the Set of All Binary Operations Over a Fixed Set

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In recent years, the word magma has been used to designate a pair of the form $(S, *)$ where $*$ is a binary operation on the set S. Inspired by that terminology, we use the notation and terminology $\mathcal{M}(S)$ (the magma of S) to denote the set of all binary operations on the set S (i.e. the set of all magmas with underlying set S.)

Our study concerns a monoid structure $(\mathcal{M}(S), \triangleleft)$ satisfying that each outset, $out(*) = \{\circ \in \mathcal{M}(S) | * \text{ distributes over } \circ\}$, is a submonoid. This endowment gives us a possibility to compare the properties of an operation $* \in \mathcal{M}(S)$ and those of the monoid structure of $(out(*), \triangleleft)$. We determine that isomorphic operations yield isomorphic outsets and explore possible converses for that result.

Several properties of $(\mathcal{M}(S), \triangleleft)$ are considered, including a complete characterization of its group of units and of a subgroup of its group of automorphisms, induced by permutations, which is a retraction. In addition, we consider various submonoids and ideals; among other results, we obtain a generic decomposition, called the cokernel-kernel decomposition, of arbitrary magmas and ideals. We also characterize those cases when a kernel-cokernel decomposition is also possible as we introduce anticommutative and pseudo-anticommutative operations.